Comparing Social Network Dynamic Operators

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1. Threshold-driven Social Network Dynamics

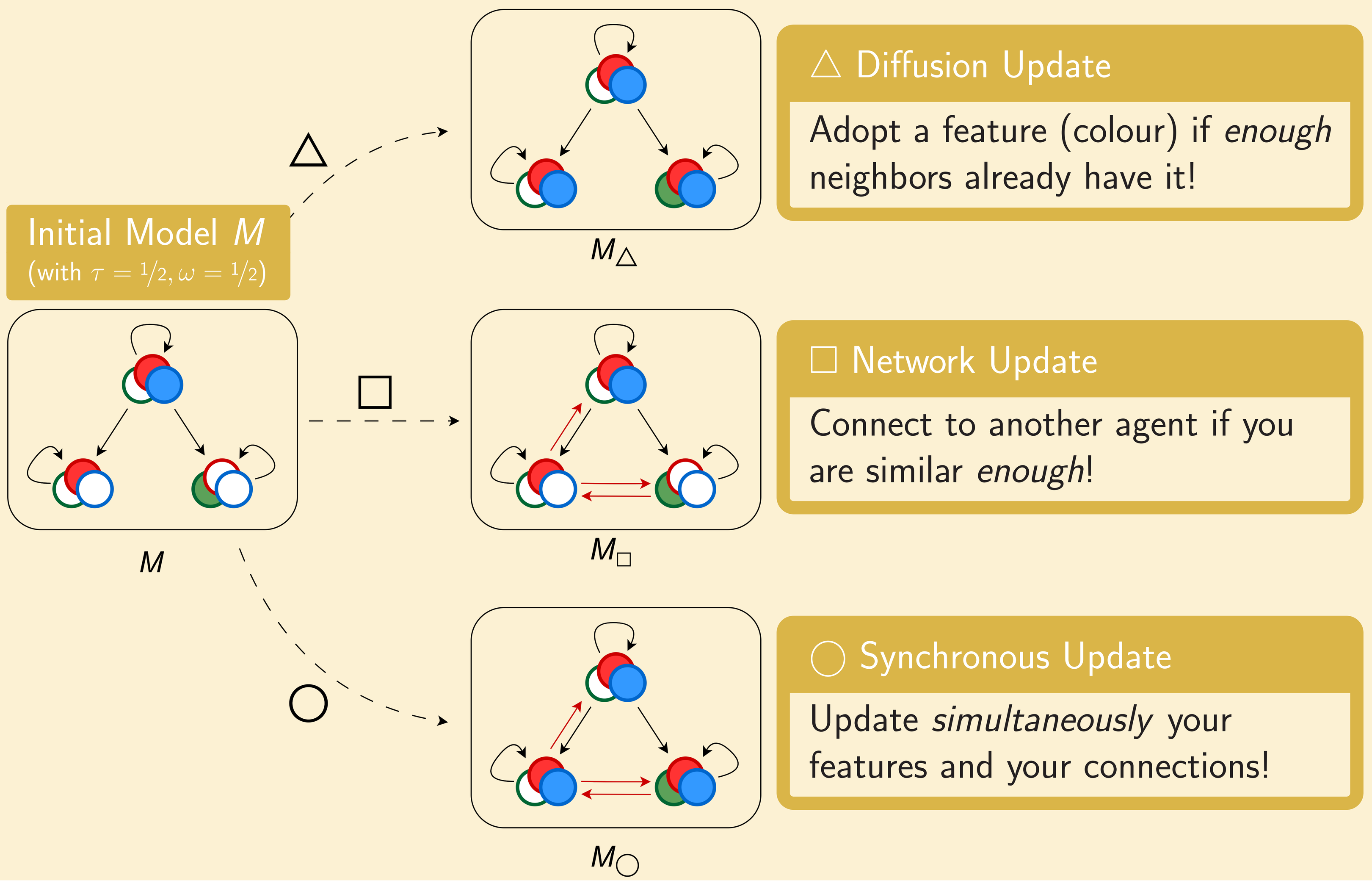
Two main processes can affect agents in a social network:

- **diffusion processes**, opinions spread from one agent to another;
- **similarity network changes**, new links form between similar agents.

Model. Let $\mathcal{A}$ be a non-empty finite set of agents, $\mathcal{F}$ be a non-empty finite set of features. A model M over $\mathcal{A}$ and $\mathcal{F}$ is a tuple $\langle \mathcal{N}, \mathcal{V}, \omega, \tau \rangle$, where:

- $\mathcal{N} \subseteq \mathcal{A} \times \mathcal{A}$ is a social influence relation;
- $\mathcal{V} : \mathcal{A} \longrightarrow \mathcal{P}(\mathcal{F})$ assigns to each agent a set of adopted features;
- $\omega \in \mathbb{Q}$ such that $0 \leq \omega \leq 1$ is the *similarity threshold*;
- $\tau \in \mathbb{Q}$ such that $0 < \tau \leq 1$ is the *influenceability threshold*.

Diffusion and network change can be **synchronous** or **asynchronous**.



2. A Logic of Asynchronous and Synchronous Changes

Syntax $\mathcal{L}$. The syntax $\mathcal{L}$ is given by

$$\varphi := N_{ab} \mid f_a \mid \neg\varphi \mid \varphi \wedge \varphi \mid \triangle\varphi \mid \square\varphi \mid \bigcirc\varphi$$

where $f \in \mathcal{F}$ and $a, b \in \mathcal{A}$.

Conformity pressure. Agent a has network pressure to adopt feature f:

$$f^\tau_{N(a)} := \bigvee_{\{G \subseteq N \subseteq A,\ N \neq \emptyset\ :\ \frac{|G|}{|N|} \geq \tau\}} \left(\bigwedge_{b \in N} N_{ba} \wedge \bigwedge_{b \notin N} \neg N_{ba} \wedge \bigwedge_{b \in G} f_b\right).$$

Similarity pressure. Agent a has similarity pressure to connect to b:

$$sim^\omega_{ab} := \bigvee_{\{E \subseteq \mathcal{F} : \frac{|E|}{|\mathcal{F}|} \geq \omega\}} \bigwedge_{f \in E} (f_a \leftrightarrow f_b).$$

Reduction axioms. Fix $\omega \in [0, 1]$ and $\tau \in (0, 1]$.

$\square N_{ab} \leftrightarrow N_{ab} \vee sim^\omega_{ab}$	$\triangle N_{ab} \leftrightarrow N_{ab}$	$\bigcirc N_{ab} \leftrightarrow N_{ab} \vee sim^\omega_{ab}$
$\square f_a \leftrightarrow f_a$	$\triangle f_a \leftrightarrow f_a \vee f^\tau_{N(a)}$	$\bigcirc f_a \leftrightarrow f_a \vee f^\tau_{N(a)}$
$\square(\varphi \wedge \psi) \leftrightarrow \square\varphi \wedge \square\psi$	$\triangle(\varphi \wedge \psi) \leftrightarrow \triangle\varphi \wedge \triangle\psi$	$\bigcirc(\varphi \wedge \psi) \leftrightarrow \bigcirc\varphi \wedge \bigcirc\psi$
$\square\neg\varphi \leftrightarrow \neg\square\varphi$	$\triangle\neg\varphi \leftrightarrow \neg\triangle\varphi$	$\bigcirc\neg\varphi \leftrightarrow \neg\bigcirc\varphi$

Logic $L^{\omega\tau}$. For fixed ω, τ, the logic $L^{\omega\tau}$ is some complete axiomatisation and derivation rules of propositional logic, with the above reduction axioms and substitution of provably equivalents.

Theorem (Completeness)

Fix $\omega \in [0, 1]$ and $\tau \in (0, 1]$. For any $\varphi \in \mathcal{L}$:

$$\models_{\mathcal{C}^{\omega\tau}} \varphi \text{ iff } \vdash_{L^{\omega\tau}} \varphi.$$

References

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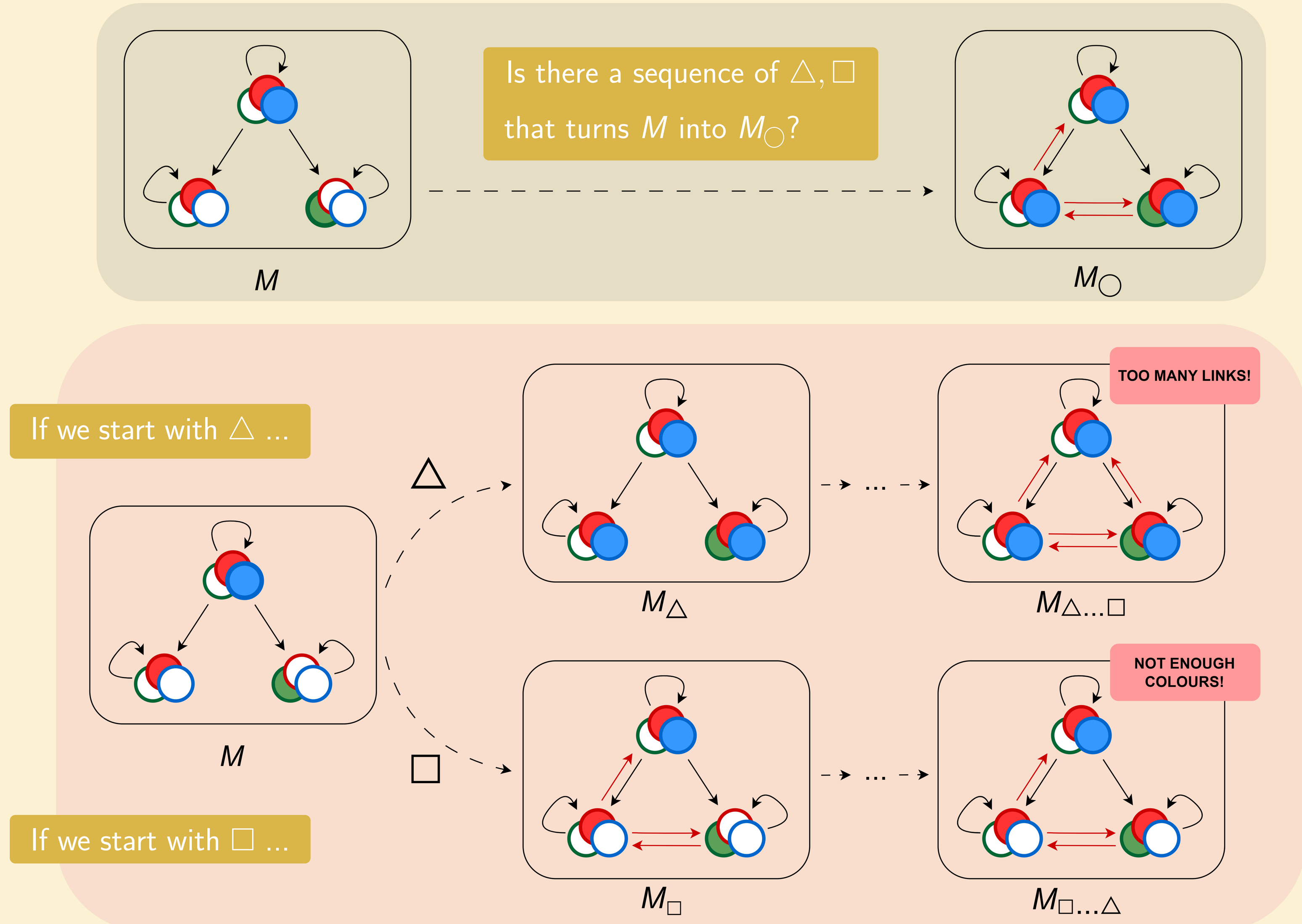
- David Easley and Jon Kleinberg. *Networks, Crowds, and Markets: Reasoning About a Highly Connected World*. Cambridge University Press, New York, USA, 2010.
- Sonja Smets and Fernando R. Velázquez-Quesada. A logical analysis of the interplay between social influence and friendship selection. In Luís Soares Barbosa and Alexandru Baltag, editors, *Dynamic Logic. New Trends and Applications*, pages 71–87, Cham, 2020. Springer International Publishing.

3. (Ir)replaceability of Synchronous Operators

QUESTION 1: Is it *always* possible to replace sequences of $\bigcirc$ (synchronous changes) with sequences of $\square, \triangle$ (asynchronous changes)?

ANSWER: No! Counterexample:



Sequences of operators. Let $D = \{\bigcirc, \triangle, \square\}$. For $O \subseteq D$, S_O denotes the set of all non-empty finite sequences of operators in O.

Equivalence of sequences. Two sequences $s_1, s_2 \in S_D$ are equivalent on a model M when for all $\varphi \in \mathcal{L}$, $M \models s_1\varphi$ if and only if $M \models s_2\varphi$.

Replaceability (in a model). Let S_1, S_2 be two sets of sequences. S_1 is replaceable with S_2 in a model M, when, for all sequences $s_1 \in S_1$, there exists $s_2 \in S_2$ equivalent to s_1 in M. S_1 is replaceable (tout court) with S_2 when it is replaceable with S_2 in all models.

Theorem ($\bigcirc$ can not *always* be replaced.)

$S_{\{\bigcirc\}}$ is not replaceable with $S_{\{\square,\triangle\}}$.

QUESTION 2: Is it *sometimes* possible to replace $\bigcirc$?

ANSWER: Yes! Exactly in these four cases:

- The network is stable.
  $\psi_\triangle := \bigwedge_{a,b \in \mathcal{A}} (N_{ab} \vee \neg sim^\omega_{ab})$
- The diffusion process is stable.
  $\psi_\square := \bigwedge_{a \in \mathcal{A}} \bigwedge_{f \in F} (f_a \vee \neg f^\tau_{N(a)})$
- A diffusion update preserves similarity between any two agents.
  $\psi_{\triangle\square} := \bigwedge_{a,b \in \mathcal{A}} (\neg N_{ab} \to (sim^\omega_{ab} \leftrightarrow \triangle sim^\omega_{ab}))$
- New links do not prevent conforming to old pressures.
  $\psi_{\square\triangle^n} := \bigwedge_{a \in \mathcal{A}} \bigwedge_{f \in F} (\neg f_a \to (f^\tau_{N(a)} \leftrightarrow \bigvee_{0 \leq i \leq n-1} \square\triangle^i f^\tau_{N(a)}))$

Theorem (When can $\bigcirc$ be replaced?)

$\bigcirc$ is replaceable by $S_{\{\triangle,\square\}}$ on a model M iff
$M \models \psi_\triangle \vee \psi_\square \vee \psi_{\triangle\square} \vee \bigvee_{0<n<|\mathcal{A}|} \psi_{\square\triangle^n}$.

Theorem (By what can $\bigcirc$ be replaced?)

If $s \in S_{\{\square,\triangle\}}$ is equivalent to $\bigcirc$ on a model M, then s is equivalent to a sequence in the set $\{\square, \triangle, \triangle\square\} \cup \{\square\triangle^n : n > 0\}$ on M.

QUESTION 3: Is it *sometimes* possible to replace $\bigcirc^m$?

ANSWER: Yes! At least in this case:

$$M \models \bigwedge_{0 \leq i \leq (m-1)} \bigcirc^i (\psi_\triangle \vee \psi_\square \vee \psi_{\triangle\square} \vee \bigvee_{0<n<|\mathcal{A}|} \psi_{\square\triangle^n}).$$